lecture 21

MTH 161

lec 21

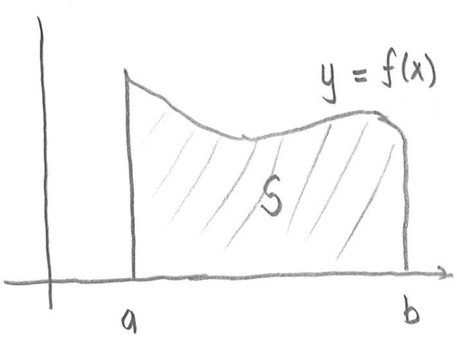
CHAPTER 4

INTEGRALS

In chapter 2, we use tangent and velocity problems to introduce the derivative, which is the central idea in differential calculus.

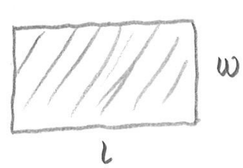
In the same way we use area and distance problems and use them to formulate the idea of definite integrals, which is the basic idea of integral calculus.

Areas and Distances

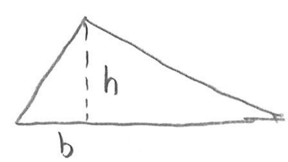


Find the area of the region S that lies under the curve $y = f(x)$ between a and b.

Difficult question



$A = lw$

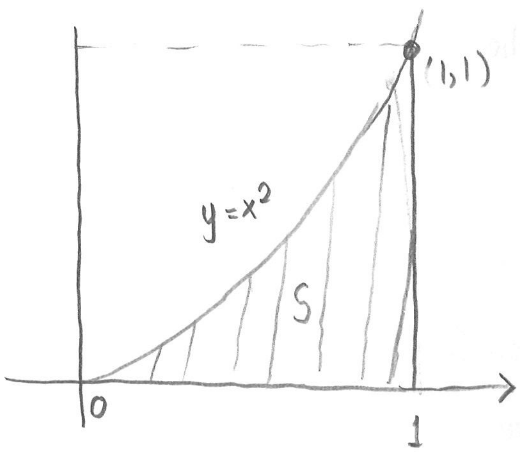


$A = \frac{1}{2} bh$

When we talked about tangent lines, we used the slopes of secant lines to approximate the slope of the tangent line, and took limit of these approximations.

We use a similar idea. We first approximate the region S by rectangles and then we take the limit of the areas of the rectangles as we increase the number of rectangles.

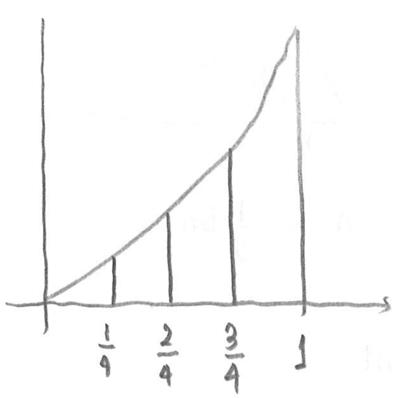
<u>Ex</u> Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

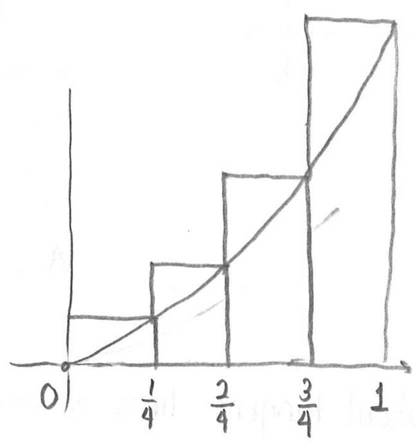


Area < 1

Divide S into four strips S_1, S_2, S_3, S_4 by drawing the vertical lines

$$x = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$$





we can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip.

So the height of each rectangle are the values of the function $f(x) = x^2$ at the right endpoints of the subintervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{2}{4}]$, $[\frac{2}{4}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$

Each rectangle has width $\frac{1}{4}$ and the heights $(\frac{1}{4})^2$, $(\frac{2}{4})^2$, $(\frac{3}{4})^2$, $(1)^2$.

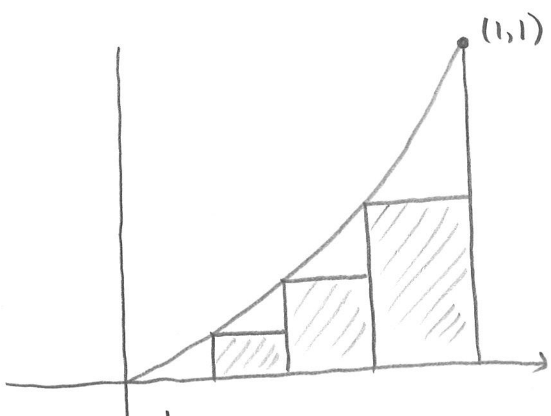
right endpoints

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = \frac{15}{32} = 0.46875$$

number of subintervals

So actually area A of S is less than R_4. Cause look at the extra area.

Next lets use the left endpoints.



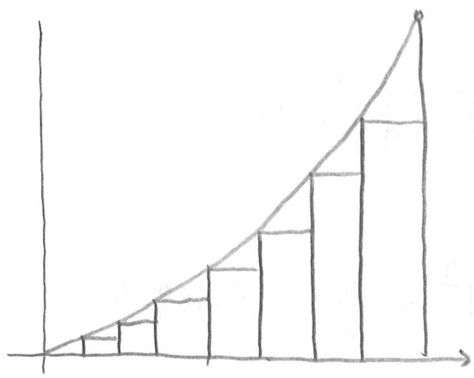
Left most rectangle has collapsed because height is zero.

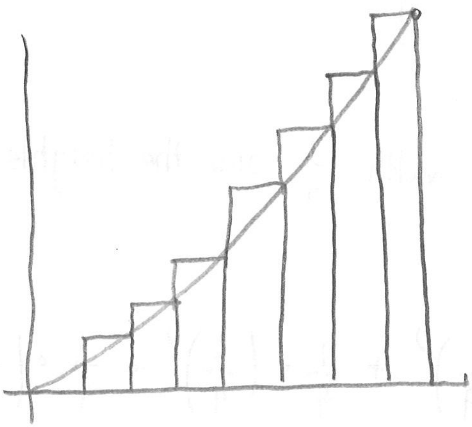
$$L_4 = 0^2 \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{2}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$$

$$= 0.21875$$

So we can say $0.21875 < A < 0.46875$

Repeat the procedure for equal 8 strips.





$L_8 = 0.2734375 < A < R_8 = 0.3984375$

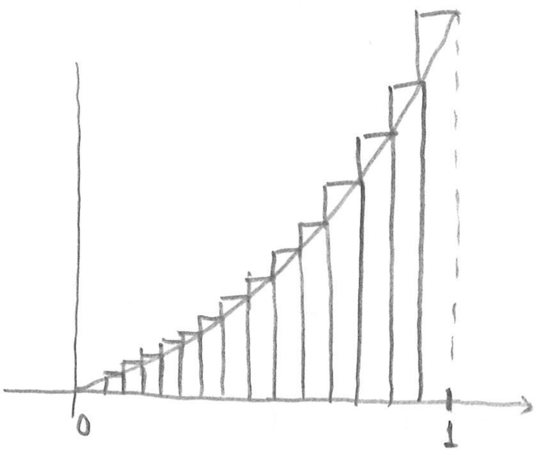
The more rectangle we take the better the estimate becomes.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

looks like R_n approaches $\frac{1}{3}$ as n-increases.

We will show,

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$



Divide $[0,1]$ into n-equal intervals

So each rectangle has width $\frac{1}{n}$.

and the heights are the values of function $f(x) = x^2$ at points $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$. So the heights are

$$\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \ldots, \left(\frac{n}{n}\right)^2$$

Thus $R_n = \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2$

$$= \frac{1}{n}\cdot\frac{(1)^2}{n^2} + \frac{1}{n}\cdot\frac{(2)^2}{n^2} + \cdots + \frac{1}{n}\cdot\frac{n^2}{n^2}$$

$$= \frac{(1)^2 + (2)^2 + \cdots + (n)^2}{n^3}$$

$$= \frac{1}{n^3}\left[(1)^2 + (2)^2 + \cdots + (n)^2\right]$$

Actually, $(1)^2 + (2)^2 + \cdots + (n)^2 = \dfrac{n(n+1)(2n+1)}{6}$

So $R_n = \dfrac{1}{n^3}\cdot\dfrac{n(n+1)(2n+1)}{6} = \dfrac{2n^2+3n+1}{6n^2}$

So, $\displaystyle\lim_{n \to \infty} = \frac{2n^2+3n+1}{6n^2}$

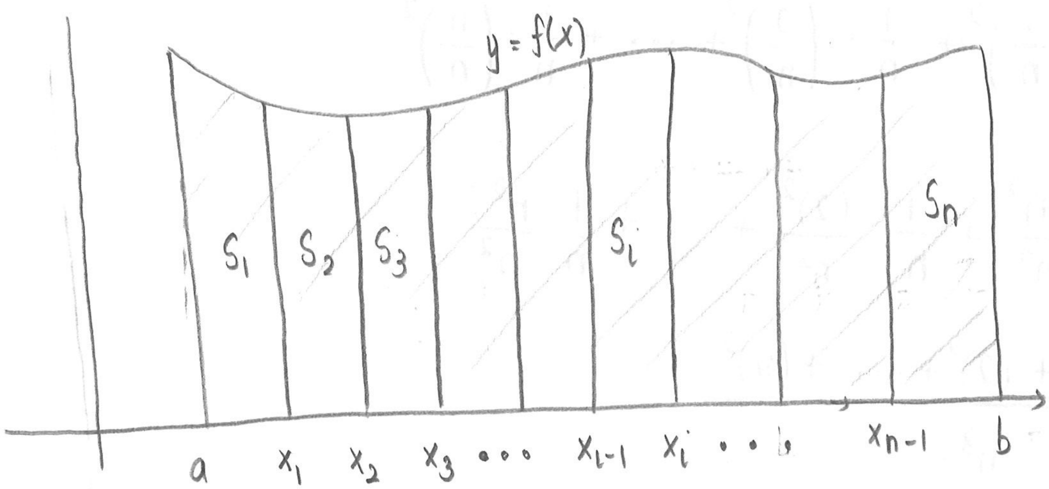
$$= \lim_{n \to \infty} \frac{\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}}{\frac{6n^2}{n^2}}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Similarly we can show that $\displaystyle\lim_{n \to \infty} L_n = \frac{1}{3}$

let's apply it to a more general case.

find area under curve $y = f(x)$ between a & b.



So we divide interval $[a, b]$ into n-intervals of equal width, so we divide S into n strips $S_1, S_2, S_3, \ldots, S_n$.

The width of the interval $[a, b]$ is $b-a$, so width of each of the n strips is

$$\Delta x = \frac{b-a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of subintervals are

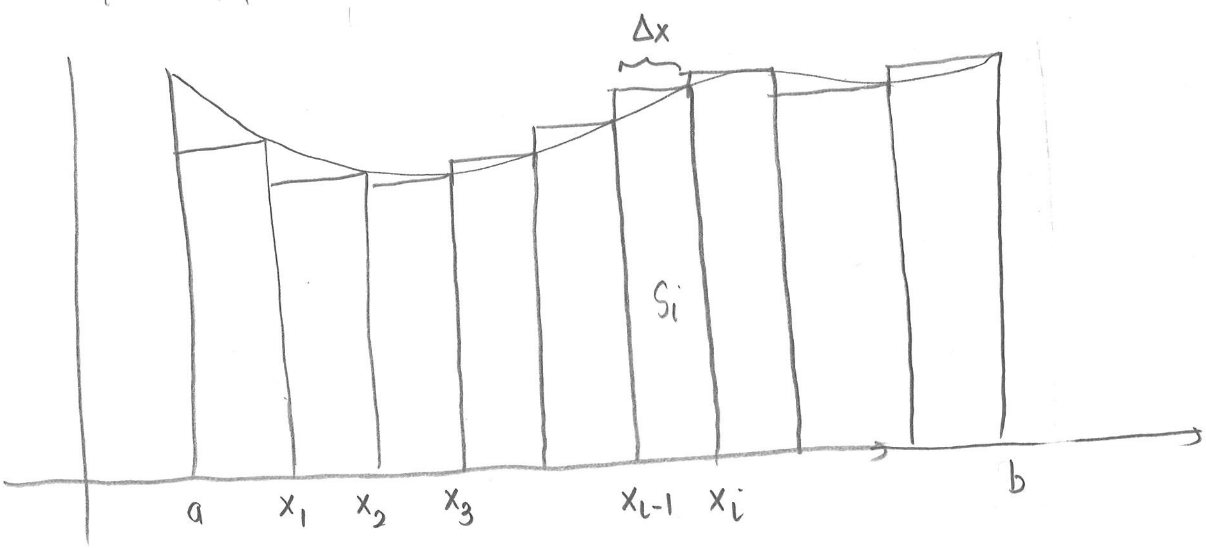
$$x_1 = x_0 + \Delta x = a + \Delta x$$

$$x_2 = x_1 + \Delta x = a + \Delta x + \Delta x = a + 2\Delta x$$

$$x_3 = x_2 + \Delta x = a + 3\Delta x$$

$$\vdots$$

$$x_i = a + i \cdot \Delta x$$



Let's approximate the i^{th} strip S_i by a rectangle with width Δx and height is $f(x_i)$. (value of the function f at the right endpoint)

So the area of the i^{th} rectangle is $f(x_i)\Delta x$

So, $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$

Defn The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x \right]$$

Remark Since the function is continuous, the limit always exists, and we get the same value when we consider the left endpoints.